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Conditional Probability on a Quantum Logic

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We analyze two approaches to conditional probability. The first approach follows Gudder and Marchand, Mączyńsky, Cassinelli and Beltrametti, Cassinelli and Truini. The second approach follows Rényi and Kalmár. The main result is a characterization of the first approach with the help of a function, similarly as in the second approach.

1. INTRODUCTION AND PRELIMINARIES

Let L be a quantum logic (i.e., orthomodular lattice) (Varadarajan, 1968): (1) L is a nonempty, partially ordered set with 0 and 1; (2) for any $\{a_n\}_{n=1}^{\infty} \subset L$, $\bigvee_n a_n$, $\bigwedge_n a_n \in L$; (3) there is a map $\perp$: L onto L such that: (a) for any $a \in L$, $(a^{\perp})^{\perp} = a$, (b) if $a \in L$, then $a \vee a^{\perp} = 1$, and (c) if a, $b \in L$ and $a \leq b$, then $b^{\perp} \leq a^{\perp}$; (4) the orthomodular law is satisfied. This means that for a, $b \in L$, $a \leq b$, $b = a \vee (a^{\perp} \wedge b)$.

Elements a, $b \in L$ are called orthogonal $(a \perp b)$ if $a \leq b^{\perp}$ (or $b \leq a^{\perp}$). Elements a, $b \in L$ are called compatible $(a \leftrightarrow b)$ if there are a_1, b_1, $c \in L$, mutually orthogonal, such that $a = a_1 \vee c$, $b = b_1 \vee c$.

We shall need the following property: If $\{a, b, c\} \subset L$ and $a \leftrightarrow b$, c, then $a_1 \vee (a_2 \wedge a_3) = (a_1 \vee a_2) \wedge (a_1 \vee a_3)$, where $a_i \in \{a, b, c\}$ for any $i = 1, 2, 3$.

A map m from L into R such that (1) $m(0) = 0$ and $m(1) = 1$, and (2) $\{a_n\}_{n=1}^{\infty} \in L$ and $a_n \perp a_t$ for any $n \neq t$ implies $m(\bigvee_n a_n) = \sum_n m(a_n)$, is called a state on L.

A support of a state m, if it exists, is an element $a \in L$ such that $m(b) = 0$ iff $a \perp b$ [briefly, $a = s(m)$].

Let M be a set of states on L. The pair (L, M) will be called a quite full system (q.f.s.) if $\{m \in M | m(a) = 1\} \subset \{m \in M | m(b) = 1\}$ implies $a \leq b$. The pair (L, M) will be called a supported system if for any $m \in M$, $s(m) \in L$ and for any $a \in L_0$(where $L_0 = L - \{0\}$) there is an $m \in M$ such that $a = s(m)$.

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It is clear that a supported system (L, M) is q.f.s. Moreover, for (L, M) a supported system, if $m(a)=m(b)=1$ (where $a, b \in L$, $m \in M$), then $m(a \wedge b)=1$.

Let us define the binary operation "$*$" on L as follows: $a*b=(a \vee b^{\perp}) \wedge b$. We shall need the following simple lemmas.

Lemma 1.1. Let $a, b \in L$. Then the following statements are equivalent:

1. $a*b=b*a$.
2. $a*b=a \wedge b$.
3. $a*b \leftrightarrow b*a$.
4. $a \leftrightarrow b$.

The proof is obvious.

Lemma 1.2. Let $a, b \in L$. Then:

1. $(a*b)^{\perp} \wedge (b*a)=0$.
2. $a*b=0$ iff $a \perp b$.
3. If $a \perp b$ and $c \in L$, then

$$(c*a) \vee (c*b)=(c \vee b^{\perp}) \wedge (c \vee a^{\perp}) \wedge (a \vee b)$$

4. $a=(b*a) \vee (b^{\perp} \wedge a)$, where $b*a^{\perp\perp}b^{\perp} \wedge a$.

The proof is obvious.

2. TWO APPROACHES TO CONDITIONAL PROBABILITY ON A QUANTUM LOGIC

In this section we analyze two approaches to conditional probability on (L, M) as a supported system.

Let (L, M) be a supported system and let M be a σ-convex set of states. For $a \in L_0$ put $M_a=\{m \in M | m(a)>0\}$. Cassinelli and Beltrametti (1975) defined a transformation Ω_a from M_a into M_a such that

I. $s(\Omega_a m)=s(m)*a$.

Cassinelli and Truini (1984) added the following properties:

II. Let $m \in M$ and put $L(m)=\{b \in L | m(b)>0\}$. Then $\Omega_{(\cdot)}m$: $L(m) \to M$ is a map. And for any $b \in L_0$, $\Omega_b(\cdot)$: $M_b \to M$ is a map.

III. If $a, b \in L$, $a \leq b$, and $m(b)>0$, then $\Omega_b m(a)=m(a)/m(b)$. Then the number $\Omega_b m(a)$ is called the conditional probability of the event a by the condition b in the state m.

In the following we suppose that the function

$$p^m(\cdot|\cdot) = \Omega_{(\cdot)}m(\cdot)\colon L\times L_c \to R,\ L_c \subset L(m)$$

satisfies the axioms I and II.

The following statements are true (Cassinelli and Beltrametti, 1975) for $b \in L(m)$:

1. $p^m(b|b) = 1$.
2. If $s(m) \geq b$, then $s(p^m(\cdot|b)) = b$.
3. If $s(m) \leq b$, then $s(p^m(\cdot|b)) = s(m)$.
4. If $a \leftrightarrow b$ and $m(a) = 1$, then $p^m(a|b) = 1$.
5. If $a \leftrightarrow b$, then $p(a|b) = p(a \wedge b|b)$.

Let us write $L^*(m) = \{b \in L(m) | b \leftrightarrow s(m)\}$.

Proposition 2.1. Let $b * c$, $b^\perp \wedge c \in L^*(m)$. Then

$$p^m(\cdot|c) = p^m(b*c|c)p^m(\cdot|b*c) + p^m(b^\perp \wedge c|c)p^m(\cdot|b^\perp \wedge c)$$

Proof. Since $b * c$, $b^\perp \wedge c \in L^*(m)$, then $c \in L^*(m)$. Moreover,

$$s(m) \wedge c = s(m) \wedge (b * c) \vee s(m) \wedge b^\perp \wedge c$$

But $s(m) \wedge c = s(p^m(\cdot|c))$, $s(m) \wedge (b*c) = s(p^m(\cdot|b*c))$, $s(m) \wedge b^\perp \wedge c = s(p^m(\cdot|b^* \wedge c))$. From these facts we have, for $q \in (0, 1)$,

$$p^m(\cdot|c) = q\cdot p^m(\cdot|b*c) + (1-q)\cdot p^m(\cdot|b^\perp \wedge c)$$

Since $p^m(b*c|c) = q\cdot p^m(b*c|b*c) = q$ and $p^m(b^\perp \wedge c|c) = 1-q$, we get

$$p^m(\cdot|c) = p^m(b*c|c)p^m(\cdot|b*c) + p^m(b^\perp \wedge c|c)p^m(\cdot|b^\perp \wedge c) \quad \blacksquare$$

Let L be a Boolean σ-algebra. Rényi (1956) defined the conditional probability as follows:

Let $L_c \subset L_0$ and $P\colon L\times L_c \to R$.

1. If $A \in L_c$, then $P(A|A) = 1$ and $P(\cdot|A)$ is a probability measure on L.
2. Let $B \in L$, C, $A \in L_c$ such that $B \subset A$; then

$$P(B|C) = P(A|C)\cdot P(B|A \cap C)$$

Then the function $P(\cdot|\cdot)$ is called a conditional probability on L.

Now it is clear that the function $p^m(\cdot|\cdot)$ on $L\times L(m) \to R$ with Axioms I and II on a Boolean α-algebra L is a conditional probability in the sense of Rényi. Moreover, if $b \in L$ (where L is a Boolean σ-algebra), $L_c \subset L(m)$, and a, $c \in L_c$, then

$$p^m(b|c) = p^m(a|c)\cdot p^m(b|a \wedge c) + p^m(a^\perp|c)\cdot p^m(b|a^\perp \wedge c)$$

Theorem 2.2. Let $m \in M$ and b, $b^{\perp} \in L(m)$. Then

$$m(\cdot) = m(b) \cdot p^m(\cdot|b) + m(b^{\perp}) \cdot p^m(\cdot|b^{\perp}) \quad \text{iff} \quad b \leftrightarrow s(m)$$

Proof. Let $b \leftrightarrow s(m)$. Then

$$\begin{aligned} s(m) &= b \wedge s(m) \vee b^{\perp} \wedge s(m) \\ &= s(p^m(\cdot|b)) \vee s(p^m(\cdot|b^{\perp})) \\ &= s(q \cdot p^m(\cdot|b) + (1-q) \cdot p^m(\cdot|b^{\perp})) \end{aligned}$$

But

$$m(b) = q \cdot p^m(b|b) = q, \qquad m(b^{\perp}) = (1-q) \cdot p^m(b^{\perp}|b^{\perp}) = 1-q$$

This means that

$$m(\cdot) = m(b) \cdot p^m(\cdot|b) + m(b^{\perp}) \cdot p^m(\cdot|b^{\perp})$$

Now we show the converse implication. Let

$$m(\cdot) = m(b) \cdot p^m(\cdot|b) + m(b^{\perp}) \cdot p^m(\cdot|b^{\perp})$$

Then

$$\begin{aligned} s(m) &= s(m) * b \vee s(m) * b^{\perp} = (s(m) \vee b^{\perp}) \wedge (s(m) \vee b) \wedge (b \vee b^{\perp}) \\ &= (s(m) \vee b^{\perp}) \wedge (s(m) \vee b)) \end{aligned}$$

This means that $s(m) \leftrightarrow b$. ■

Corollary 2.2.1. If $L_c = L^*(m)$, then for a, $b \in L$, $a \leq b$, $b \in L_c$

$$p^m(a|b) = m(a)/m(b)$$

In other words, there is an analogy with the Kolmogoroff conditional probability. Indeed, if $p: L \times L_c \to R$, where $L_c \subset L^*(m)$, Axiom III of Cassinelli and Truini (1984) is satisfied.

Now we compare Kalmár's definition of conditional probability with the Cassinelli–Truini approach.

Definition 2.1. (Kalmár, 1983). Let L be a quantum logic and $L_c \subset L_0$. If p: $L \times L_c \to [0, \infty)$ with (A) $p(\cdot|b)$ is a state on L for all $b \in L_c$, and (B) for all $b \in L_c$, $p(b|b) = 1$; then the function $p(\cdot|\cdot)$ is called a conditional probability on L.

This definition is very general. Indeed, on a Boolean σ-algebra it is more general then the classical definition (Kolmogoroff, 1933; Rényi, 1956). Hence, Kalmár added the following axioms:

(C) If a, $b \in L$, c, $c * b \in L_c$, $a \leftrightarrow b$, then

$$p(a \wedge b|c) = p(b|c) \cdot p(a|c * b)$$

(D) If $a \vee b \in L_c$, then $p(a|a \vee b) + p(b|a \vee b) > 0$.

(E) If $(a \vee b) \wedge (a \vee b) \in L_c$, then

$$p(a|(a \vee b) \wedge (a \vee b^{\perp})) > 0$$

Proposition 2.3. Let (L, M) be a supported system and M be a σ-convex set of states. If $m \in M$, $L_c \subset L(m)$, and $a \vee b \in L_c$, then

$$p^m(a|a \vee b) + p^m(b|a \vee b) > 0$$

Proof. Let $p^m(a|a \vee b) = p^m(b|a \vee b) = 0$. Then $1 = p^m(a^{\perp} \wedge b^{\perp}|a \vee b)$. This means that

$$s(m) * (a \vee b) \leq (a^{\perp} \wedge b^{\perp}) \wedge (a \vee b) = 0$$

From this fact we have $s(m) * (a \vee b) = 0$ and then $a \vee b \notin L_c$. Then $p^m(a|a \vee b) + p^m(b|a \vee b) > 0$. ■

In the following we assume that (L, M) is a supported system and M is a σ-convex set of states.

Example 2.1. Let $m \in M$, $L_c \in L(m)$. Let us write $a^{\perp} = s(m)$. Now we choose $b \in L$ such that $b \not\leftrightarrow a$ and $a \vee b = a \vee b^{\perp} = 1$ (such an element b exists if, for example, $a^{\perp}$ is an atom in L). Then $(a \vee b) \wedge (a \vee b^{\perp}) = 1$ and $s(p^m(\cdot|(a \vee b) \wedge (a \vee b^{\perp}) = s(m)$. But $m(a) = 0$.

Proposition 2.4. Let $m \in M$ and $(a \vee b) \wedge (a \vee b^{\perp}) \in L_c$, where $L_c \subset L(m)$. Then:

1. If $p^m(a|(a \vee b) \wedge (a \vee b^{\perp})) = 0$, then $s(m) \wedge a = 0$.
2. If $s(m) = 1$, then $p^m(a|(a \vee b) \wedge (a \vee b^{\perp})) > 0$.

Proof. (1) If $p^m(a|(a \vee b) \wedge (a \vee b^{\perp})) = 0$, then $s(m) * ((a \vee b) \wedge (a \vee b^{\perp})) \leq a$. It means $s(m) \wedge a \leq a^{\perp} = 0$.

Statement 2 follows from 1. ■

The author does not know if Axiom C is satisfied.

3. THE CONDITIONAL PROBABILITY ON QUANTUM LOGIC AS A FUNCTION

Let (L, M) be q.f.s. and M be a σ-convex set of states. For simplicity of formulation we shall assume that $L_c \subset L_0$ and $1 \in L_c$.

Definition 3.1. Let (L, M) be a q.f.s. and M be a σ-convex set of states. Let $p: L \times L_c \to [0, \infty)$ satisfy:

(a) For any $b \in L_c$, $p(\cdot|b) \in M$ and $p(b|b) = 1$.

(b) If $p(c|1) = 1$ and b, $c * b \in L_c$, then

$$p(\cdot|b) = p(c * b|b) \cdot p(\cdot|c * b)$$

(c) If $a, b, c \in L$ and $a \vee b$, $a \vee b \vee c \in L_c$, then

$$p(a|a \vee b \vee c) = p(a|a \vee b) \cdot p(a \vee b|a \vee b \vee c)$$

The function $p(\cdot|\cdot)$ will be called a function of conditional probability on L.

Lemma 3.1. Let $a, b, c \in L$. Then:

1. If $b \in L_c$, then $p(a|b) = p(b * a|b)$.
2. If $a \leftrightarrow b$, $b \in L_c$, then $p(a|b) = p(a \wedge b|b)$.
3. If b, $b \vee c \in L_c$ and $a \leq b$, then $p(a|b \vee c) = p(a|b) \cdot p(b|b \vee c)$.

The proof of this Lemma is obvious.

The main result of this paper is summarized in the following theorem.

Theorem 3.2. Let (L, M) be q.f.s. Let M be a σ-convex set of states. Let $m \in M$ and $a, b \in L$ such that $m(a) = m(b) = 1$ holds, and $m(a \wedge b) = 1$. Let $c \in L_0$ and $L_c = \{b \in L | b \not\perp c\}$. Let $p : L \times L_c \to [0, 1]$ satisfy Axioms (a)–(c) of Definition 3.1. Let there be $s(p(\cdot|1)) = c$. Then for any $b \in L_c$, $s(p(\cdot|b))$ exists and $s(p(\cdot|b)) = c * b$. Moreover, if $a \leq b$ $(a \in L,\ b \in L_c)$, then

$$p(a|b) = p(a|1)/p(b|1)$$

Conversely, let (L, M) be a supported system and M be a σ-convex set of states. If the function $p^m(\cdot|\cdot)$, $m \in M$, satisfies the axioms I–III on $L \times L(m)$, then it satisfies the axioms (a)–(c) of Definition 3.1.

Proof. First we show that $s(p(\cdot|b) = c * b$ for all $b \in L_c$. This means that we must prove that $p(a|b) = 0$ iff $a \perp c * b$. Let $p(a|b) = 0$. Then $p(a^\perp \wedge b|b) = 1$. Hence

$$\begin{aligned} 0 &= p(b \wedge (a^\perp \wedge b)^\perp|b) = p(b \wedge (a^\perp \wedge b)^\perp|b) \cdot p(b|1) \\ &= p(b \wedge (a^\perp \wedge b)|1) \end{aligned}$$

The last equality holds by Lemma 3.1, part 3. But $s(p(\cdot|1)) = c$. Then $c \perp b \wedge (a^\perp \wedge b)^\perp$. This means that

$$c \vee b^\perp \vee (a^\perp \wedge b) = b^\perp \vee (a^\perp \wedge b) \tag{1}$$

Now we have

$$\begin{aligned} (c * b) \vee b^\perp \vee (a^\perp \wedge b) &= ((c \vee b^\perp) \wedge b) \vee b^\perp \vee (a^\perp \wedge b) \\ &= ((c \vee b^\perp) \wedge (b \vee b^\perp)) \vee (a^\perp \wedge b) \\ &= (c \vee b^\perp) \vee (a^\perp \wedge b) \\ &= c \vee b^\perp \vee (a^\perp \wedge b) \\ &= b^\perp \vee (a^\perp \wedge b) \end{aligned}$$

The last equality follows from (1) and this implies that $c*b \leq b^\perp \vee (a^\perp \wedge b)$. Moreover, $c*b \leq b$. Then

$$c*b \leq (b^\perp \vee (a^\perp \wedge b)) \wedge b = a^\perp \wedge b \leq a^\perp \tag{2}$$

From relation (2) we have $c*b \perp a$.

Let $c*b \perp a$. Since $p(c|1)=1$, we can use Axiom (b) from Definition 3.1. We get

$$p(a|b) = p(c*b|1) \cdot p(a|c*b)$$

But $p(a|c*b)=0$. This means that $s(p(\cdot|b))$ exists and $s(p(\cdot|b)) = c*b$.

Let $a \leqslant b$. Then from Lemma 3.1, part 3 we have

$$\frac{p(a|1)}{p(b|1)} = \frac{p(a|b)\cdot p(b|1)}{p(b|1)} = p(a|b)$$

It is clear that the function $p^m(\cdot|\cdot)$ satisfies Axiom (a). Axiom (c) follows directly from Axiom III. We shall show only (b).

Let $p^m(c|1)=1$. Then $s(m) \leq c$. Since

$$p^m(c*b|b) = m(c*b)/m(b) = m(b)/m(b) = 1$$

then $p^m(c*b|b)\cdot p^m(\cdot|c*b) = p^m(\cdot|c*b)$. But

$$\begin{aligned} s(p^m(\cdot|c*b)) &= s(m)*(c*b) \\ &= (s(m) \vee (c*b)^\perp) \wedge (c*b) \\ &= (s(m) \vee (c^\perp \wedge b) \vee b^\perp) \wedge (c \vee b^\perp) \wedge b \\ &= (s(m) \vee b^\perp \vee (c^\perp \wedge b)) \wedge (c \vee b^\perp) \wedge b \\ &= (s(m) \vee b^\perp) \wedge b \\ &= s(m)*b \\ &= s(p^m(\cdot|b)) \end{aligned}$$

From this we have $p^m(\cdot|b) = p^m(c*b|b)\cdot p^m(\cdot|c*b)$. ■

Corollary 3.2.1. Let m be from M and $s(p(\cdot|1)) = s(m)$. Then for $a \in L$, $a \leq b$ $[b \in L(m)]$ we have

$$p(a|b) = m(a)/m(b)$$

Theorem 3.3. Let (L, M) be a supported system and let M be a σ-convex set of states. If $s(p(\cdot|1)=1$, then

$$p(a|b) = p(a \wedge b|b) \qquad \text{iff} \quad a \leftrightarrow b$$

Proof. It is enough to show that $p(a|b) = p(a \wedge b|b)$ implies $a \leftrightarrow b$.

Let $p(a|b)=p(a \wedge b|b)$. This means that $0=p(a \wedge (a \wedge b)^{\perp}|b)=p(b^{\perp}/b)$. Now we have

$$0=p(b^{\perp} \vee (a \wedge (a \wedge b)^{\perp})|b)=p((b^{\perp} \vee a) \wedge (a| \wedge b)^{\perp}|b)$$

But $s(p(\cdot|b))=b$ and then we get $b \perp (b^{\perp} \vee a) \wedge (a \wedge b^{\perp})$. This means that

$$b^{\perp} \leq (b^{\perp} \vee a^{\perp}) \wedge (b^{\perp} \vee a) \leq b^{\perp}$$

But from the property of compatibility we have $a \leftrightarrow b$. ■

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